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A Note on Band Anticongruence of Ordered Semigroups

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Abstract

Some conditions on existence of band anti-congruence of orderd semigroups under a pair of order and anti-order in Bishop's constructive mathematics are presented.

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1 Preliminaries and Introduction

Our setting is Bishop's constructive mathematics (in the sense of [1], [2] and [6]), mathematics developed with Constructive Logic (or Intuitionistic Logic ([11])) - logic without the Law of Excluded Middle $P \vee \neg P$. We have to note that 'the crazy axiom' $\neg P \Longrightarrow (P \Longrightarrow Q)$ is included in Constructive Logic. Precisely, in Constructive logic the 'Double Negation Law' $P \Longleftrightarrow \neg\neg P$ does not hold but the following implication $P \Longrightarrow \neg\neg P$ holds even in Minimal Logic. In Constructive Logic 'Weak Law of Excluded Middle' $\neg P \vee \neg\neg P$ does not hold, too. It is interesting that in Constructive logic the following deduction principle $A \vee B, \neg A \vdash B$ holds, but we are not able to prove it without 'the crazy axiom'.

Any notion in Bishop's constructive mathematics has positively defined symmetrical pair since Law of Excluded Middle does not hold in Constructive Logic. Our intention is development of these symmetrical notions and investigate their compatibility with so-called the 'first notions' in Semigroup Theory. As the first, semigroup is equipped with diversity relation compatible with the equality, and, the second, the semigroup internal operation is total extensional and strongly extensional mapping.

Let $(S, =, \neq)$ be a set (in the sense of [1], [2], and [6]), where " $=$ " is an equality and " $\neq$ " is a binary relation on S which satisfies the following properties:

$$(x \neq x), x \neq y \Longrightarrow y \neq x, x \neq y \wedge y = z \Longrightarrow x \cong z,$$

called *diversity relation* on S. Following Heyting, if the relation satisfies the following implication

$$x \neq z \Longrightarrow (\forall y \in S)(x \neq y \vee y \neq z),$$

we say that it is an *apartness*. Let Y be a subset of S and let $x \in S$. Following Bridges, by $x \bowtie Y$ we denote $(\forall y \in Y)(y \neq x)$ and by Y^C we denote the subset $x \in S : x \bowtie Y$ - the strong complement of Y in S ([11]). The subset Y of S is *strongly extensional* ([11]) in S if and only if $y \in Y \Longrightarrow y \neq x \vee x \in Y$.

Let S be a set with apartness and let α and β be relations on S. The *filed product* ([7-10]) of α and β is the relation defined by $\beta * \alpha = \{(x, z) \in S \times S : (\forall y \in S)((x, y) \in \alpha \vee (y, z) \in \beta)\}$. For $n \geq 2$, let ${}^n\alpha = \alpha * \ * \alpha$ (n factors). Put ${}^1\alpha = \alpha$. By $c(\alpha)$, we denote the intersection $c(\alpha) = \cap^n_{n \in N} \alpha$. The relation $c(\alpha)$ is a cotransitive relation on S, by Theorem 0.4 of [9], called *cotransitive internal fulfillment* of the relation α.

A relation q on S is a *coequality relation* on S ([9], [10]) if and only if q (consistensy), $q^{-1} = q$ (symmetry) and $q \subseteq q * q$ (cotransitivity).

In this case we can construct the following factor set $S/q = \{aq : a \in S\}$ with:

$$aq =_1 bq \Longleftrightarrow (a,b) \bowtie q, aq \neq_1 bq \Longleftrightarrow (a,b) \in q.$$

Let $S = (S, =, \neq, \cdot)$ be a semigroup with apartness and the semigroup operation being strongly extensional in the following sense

$$(\forall a,b,x,y \in S)((ay \neq by \Longrightarrow a \neq b) \wedge (xa \neq xb \Longrightarrow a \cong b)).$$

Recall that a semigroup S is called *band* if $a^2 = a$, for all $a \in S$. A subset T of S is a *completely prime* subset of S ([3]) if and only if

$$(\forall x,y \in S)(xyinT \Longrightarrow x \in T \vee y \in T).$$

Let q be a coequality relation on semigroup S. For q we say that it is *anti-congruence* on S ([9], [10]) if and only if

$$(\forall a,b,x,y \in S)((ax,by) \in q \Longrightarrow (a,b) \in q \vee (x,y) \in q).$$

This is equivalent with the following:

$$(\forall u,x,y \in S)(((ux,uy) \in q \Longrightarrow (a,b) \in q) \wedge ((xu,yu) \in q \Longrightarrow (x,y) \in q)).$$

If q is anti-congruence on semigroup S, then the strong complement q^C of q is a congruence on the semigroup S compatible with q. (For equality e and coequality q on a semigroup S we say that they are *compatible* if and only if $qoe \subseteq q$ and $eoq \subseteq q$.) We can construct semigroups $S/(q^C, q) = \{aq^C : a \in S\}$ and S/q with

$$aq^C =_1 bq^C (a,b) \bowtie q, aq^C \neq_1 bq^C \Longleftrightarrow (a,b) \in q, aq^C \cdot bq^C = (ab)q^C.$$

$$aq =_1 bq \Longleftrightarrow (a,b) \bowtie q, aq \neq_1 bq \Longleftrightarrow (a,b) \in q, aq \cdot bq = (ab)q.$$

It is easy to establish the fact: $S/(qC, q) \cong S/q$.

There is a very interesting property of coequality relation on semigroup S with apartness ([10], Theorem 5): Let q be a coequality relation on a semigroup S with apartness. Then the relation $q^+ = \{(x,y) \in S \times S : (\exists a,b \in S^1)((axb, ayb)q)\}$ is an anti-congruence on S and it is minimal extension of q.

Let us recall some standard notions and notations about relations and mappings: For relation $\theta \subseteq S \times S$ we say ([8]) that it is an *anti-order relation* on semigroup S if and only if:

$$\theta \subseteq \neq, \neq \subseteq \theta \cup \theta^{-1} (linearity), \theta \subseteq \theta * \theta$$

and compatible with the semigroup operation:

$$(\forall a, b, x, y \in S)(((ay, by) \in \theta \Longrightarrow (a, b) \in \theta) \wedge ((xa, xb) \in \theta \Longrightarrow (a, b) \in \theta)).$$

Relations $\leq$ and θ are *compatible* if and only if $\neg(x \leq y \wedge x\theta y)$. A mapping $\varphi : S \longrightarrow T$ must be *strongly extensional*: $(\forall x, x' \in S)(\varphi(x) \neq_T \varphi(x') \Longrightarrow x \neq_S x')$; φ is an *embedding* if and only if $(\forall x, x' \in S)(x \neq_S x' \Longrightarrow \varphi(x) \neq_T \varphi(x'))$. If $\varphi : S \longrightarrow T$ is a strongly extensional mapping between sets with apartnesses, then the sets $Ker\varphi = \{(x, x') \in S \times S : \varphi(x) =_T \varphi(x')\}$ and $Anti - ker\varphi = \{(x, x') \in S \times S : \varphi(x) \neq_T \varphi(x')\}$ are compatible equality an coequality relation on S. Also, for the mapping $f : (S, \leq, \theta) \longrightarrow (T, \leq, \Theta)$ we say that it is *order isotone* if $x \leq y \Longrightarrow f(x) \leq f(y)$ holds; f is *order reverse isotone* if $f(x) \leq f(y) \Longrightarrow x \leq y$; f is *anti-order isotone* if $x\theta y \Longrightarrow f(x)\Theta f(y)$ holds; f is *anti-order reverse isotone* if $f(x)\Theta f(y) \Longrightarrow x\theta y$ holds.

Example: Let T be a subset of a semigroup S. Then:

(1) The relation q on S defined by $(a, b) \in Q \Longleftrightarrow (\exists x, yS^1)((xay \in T \wedge xby \bowtie T) \vee (xby \in T \wedge xay \bowtie T))$ is an anti-congruence on S.

(2) For a subset T of a semigroup S we say that it is a *consistent subset* of S if and only if $(\forall x, y \in S)(xy \in T \Longrightarrow x \in T \wedge y \in T)$ holds. If T is a consistent subset of S, then the relation q on S, defined by $(a, b) \in q \Longleftrightarrow a \neq b \wedge (a \in T \vee b \in T)$, is an anti-congruence on S.

Semigroups with apartnesses were defined and studied for the first time by A.Heyting. P.T.Johnstone, J.C.Mulvey, F.Richman, R.Mines, D.A.Romano, W.Ruitenburg, A.S.Troelstra and D.van Dalen also have some results in this field. There are more general problems on semigroup with apartness in Constructive Algebra. In this paper we give a construction of a coequality relation q on an ordered semigroup $(S, =, \neq, \cdot, \leq, \theta)$ with apartness under a pair of an order "$\leq$" and an anti-order "θ" relations such that q is a band anti-congruence on ordered semigroup S. (Definition of band anti-congruence on semigroup S ordered under a pair of relations is given below.)

For undefined notions and notations of Semigroup Theory we refer to [3], [4] and of items in Constructive Mathematics we refer to [1-2], [6] and [11] and to [8-10].

Lemma 0 ([3], Theorem 1.24) *The following conditions for a semigroup S are equivalent*:

(1) $(\forall a, b \in S)((a, ab) \bowtie q \wedge (a, ba) \bowtie q)$;
(2) $(\forall a, b \in S)((a, aba) \bowtie q)$;
(3) $(\forall a, b, c \in S)((a, a^2) \bowtie q \wedge (abc, ac) \bowtie q)$.

We have the following:

Lemma 1: Relation θ on band S, defined by

$$(a, b) \Longleftrightarrow a \neq ab \vee a \neq ba$$

is an anti-order relation on S.

Proof: $a \neq b \Longrightarrow a \neq ab \vee ab \neq b$
$\Longrightarrow (a, b) \in \theta \vee (b, a) \in \theta$;
$(a, c) \in \theta \Longleftrightarrow a \neq ac \vee a \neq ca$
$\Longrightarrow (a \neq ab \vee ab \neq abc \vee abc \neq ac) \vee (a \neq ba \vee ba \neq cba \vee cba \neq ca)$
$\Longrightarrow (a \neq ab \vee b \neq bc \vee ab \neq a) \vee (a \neq ba \vee b \neq cb \vee cb \neq c)$
$\Longrightarrow (a, b) \in \theta \vee (b, c) \in \theta$;
$(a, b) \in \theta \Longleftrightarrow a \neq ab \vee a \neq ba$
$\Longrightarrow a \neq a^2 \vee a^2 \neq ab \vee a \neq a2 \vee a2 \neq ba$
$\Longrightarrow a \neq b$. $\Box$

So, band S is supplied with compatible pair of relations: order "$\leq$" and anti-order "θ" defined by $x \leq y$ iff $x = xy = yx$, and $x\theta y$ iff $x \neq xy$ or $x \neq yx$, for every x,y in S. Let us note that if $(S, =, \neq)$ is a band, then $(S, =, \neq, \cdot, \leq, \theta)$ is not an ordered semigroup, in general (unless in the special case when the multiplication on S is commutative).

If q^C is a band congruence on semigroup $S = (S, =, \neq, \cdot)$, i.e. if

$$(\forall a \in S)((a, a^2) \bowtie q),$$

then we say that q is a *band anti-congruence* on S. If q^C is a left zero and right zero band congruence on S ([3] [4]), i.e. if $(\forall a, b \in S)((a, ab) \bowtie q \wedge (a, ba) \bowtie q)$ we say that q is a *rectangular band anti-congruence* on S.

If $(S, =, \neq, \cdot, \leq)$ is an ordered semigroup and $H \subseteq S$, we denote by $(H]$ the subset of S defined by $(H] = \{t \in S : (\exists h \in H)(t \leq h)\}$.

Now, following the classical definition in [5], we will define band anti-congruence on semigroup $(S, =, \neq, \cdot, \leq, \theta)$ ordered under a pair of (compatible) order "$\leq$" and anti-order relation "θ": An anti-congruence q on ordered semigroup $(S, =, \neq, \cdot, \leq, \theta)$ is a *band anti-congruence* on S if and only if

$$a \leq b \Longrightarrow (a, ab) \bowtie q \wedge (a, ba) \bowtie q,$$

and

$$(a, ab) \in q \vee (a, ba) \in q \Longrightarrow (a, b) \in \theta.$$

As it is easily seen, our definition is different from the classical definition given in [5], because we are in Constructive mathematics and, besides, must be some connection between anti-congruence and anti-order relation in such a semigroup.

2 The Main Results

In the first result we describe classes of a band anti-congruence q on an ordered semigroup:

Theorem 2 *Let* $(S, =, \neq, \cdot, \leq, \theta)$ *be an ordered semigroup and* q *a band anti-congruence on* S*. Then we have the following*:

(1) $(\forall x \in S)((x, x^2) \bowtie q \wedge (x^2, x) \bowtie q)$.

(2) S/q *is a band.*

(3) *The class* xq*, generated by the element* x *of* S*, is a strongly extensional completely prime proper subset of* S *for all* x *in* S*.*

(4) *If* $x \leq y$*, then* $(xy, yx) \bowtie q$*.*

(5) *The natural epimorphism* $\pi : S \longrightarrow S/q$ *is order isotone and anti-order reverse isotone homomorphism.*

(6) $t \bowtie xq \wedge y \in S \Longrightarrow ((ty \bowtie xq \wedge yt \bowtie xq) \vee t\theta y)$.

Proof:

(1) $(\forall x \in S)((x, x^2) \bowtie q \wedge (x^2, x) \bowtie q)$ and $(\forall x \in S)(xq \cdot xq =_1 xq)$. In fact: If $x \in S$, then $x \leq x$, and $(x, x^2) \bowtie q$. Since q is symmetric, we also have $(x^2, x) \bowtie q$. Therefore, immediately, $(\forall x \in S)(x^2 q =_1 xq)$ holds.

(2) S/q is a band. Indeed, as we have already seen, $(S/q, =_1, \neq_1, \cdot)$ is a semigroup, and by (1), it is a band.

(3) The class xq, generated by the element x of S, is a strongly extensional completely prime right subset of S for all x in S. Indeed, let $x \in S$. Clearly, $xq \subset S$ because $x \bowtie xq$. Let $uv \in xq$. Then, $(x, uv) \in q$. Thus, $(x, x^2) \in q$ or $(x^2, uv) \in q$. Since the case $(x, x^2) \in q$ is impossible, we have $(x, u) \in q$ or $(x, v) \in q$, because q is an anti-congruence on S.

(4) If $x \leq y$, then $(xy, yx) \bowtie q$. Let $x \leq y$. Since q is a band anti-congruence on S, we have $(x, xy) \bowtie q$ and $(x, yx) \bowtie q$. Let (u, v) be an arbitrary element of q. Then $(u, xy) \in q$ or $(xy, x) \in q$ or $(x, yx) \in q$ or $(yx, v) \in q$. Hence, $u \neq ab$ or $ba \neq v$, because the cases $(xy, x) \in q$ and $(x, yx) \in q$ are impossible. So, $(u, v) \neq (xy, yx)$.

(5) If q is a band anti-congruence on ordered semigroup $(S, =, \neq, \cdot, \leq, \theta)$ then, as we have already seen, $(S/q, =_1, \neq_1, \cdot)$ is a band. So, the set S/q, with the relations "$\leq_1$" and "θ_1" on S/q defined by:

$$xq \leq_1 yq \Longleftrightarrow xq =_1 xq \wedge yqxq =_1 yq \cdot xq,$$

$$xq\theta_1 yq \Longleftrightarrow xq \neq_1 xq \cdot yq \vee xq \neq_1 yq \cdot xq,$$

is ordered set under compatible order and anti-order relation. Moreover, since q is a band anti-congruence, we have

$x \leq y \Longrightarrow (x, xy) \bowtie q \wedge (x, yx) \bowtie q$

$\Longleftrightarrow xq =_1 xyq \wedge xq =_1 yxq$

$\Longrightarrow xq \leq_1 yq;$

$xq\theta_1 yq \Longleftrightarrow xq \neq_1 xq \cdot yq \vee xq \neq_1 yq \cdot xq$

$\Longleftrightarrow (x, xy) \in q \vee (x, yx) \in q$

$\Longrightarrow x\theta y.$

Therefore, the natural epimorphism $\pi : S \longrightarrow S/q$ is order isotone and anti-order reverse isotone homomorphism.

(6) $t \bowtie xq \wedge y \in S \Longrightarrow ((ty \bowtie xq \wedge yt \bowtie xq) \vee t\theta y)$. In fact: Let u be an arbitrary element of xq. Then $(u, x) \in q$. Thus $(u, ty) \in q$ or $(ty, t) \in q$ or $(t, x) \in q$. Hence, $u \neq ty$ or $t\theta y$, because $(t, x) \in q$ is impossible. So, the implication

$$t \bowtie xq \wedge y \in S \Longrightarrow (ty \bowtie xq \vee t\theta y)$$

holds. Analogously we show the implication $t \bowtie xq \wedge y \in S \Longrightarrow (yt \bowtie xq) \vee t\theta y)$. $\square$

The following theorem is the main result of our paper:

Theorem 3 *Let $(S, =, \neq, \cdot, \leq, \theta)$ be an ordered semigroup. The following are equivalent*:

(1) *There exists a band anti-congruence on S.*

(2) *There exists a band $(B, =_1, \neq_1, o, \leq_1, \theta_1)$ and a mapping $\pi : S \longrightarrow B$ which is strongly extensional order isotone and anti-order reverse isotone surjective homomorphism such that $\pi^{-1}(a)$ is a strongly extensional subsemigroup of S and the following implication $t \in \pi^{-1}(a) \wedge y \in S \Longrightarrow ((ty \in \pi^{-1}(a) \wedge yt \in \pi^{-1}(a)) \vee t\theta y)$ holds for every $a \in B$.*

(3) *There exists a band* $(B, =_1, \neq, \circ)$ *and a family* $\Re = \{S_b : b \in B\}$ *of strongly extensional subsemigroups of* S*, such that*

(3.1) $S_a \cap S_b = \emptyset$ *for all* $a, b \in B$, $a \neq_1 b$;

(3.2) $S = \cup_{b \in B} S_b$;

(3.3) $S_a S_b \subseteq S_{a \circ b}$ *for all* $a, b \in B$;

(3.4) *If* $a, b \in B$ *such that* $Sa \cap (Sb] \neq \emptyset$ *, then* $a = aob = boa$;

(3.5) $t \in Sa \wedge y \in S \Longrightarrow ((ty \in S_a \wedge yt \in S_a) \vee t\theta y)$ *for every* a *of* S.

Proof:

(1) $\Longrightarrow$ (2). Let q be a band anti-congruence on S. Then the class $aq = xq$, generated by element x, is a strongly extensional completely prime subset of S. As we have already seen $(S/q, =_1, \neq_1, o, \leq_1, \theta_1)$ is a band. We consider the mapping $\pi : S \longrightarrow S/q$ by $\pi(x) = xq$. The mapping π is a strongly extensional homomorphism. Indeed, if $x, y \in S$, then $\pi(xy) = (xy)q =_1 xqoyq = \pi(x)o\pi(y)$. Since q is a band anti-congruence on S then: if $x \leq y$, then $xq \leq_1 yq$ and if $xq\theta_1 yq$ then $x\theta y$ hold. The mapping π is clearly onto. Therefore,π is a strongly extensional order isotone and anti-order reverse isotone surjective homomorphism. Let now $x \in S$. Suppose that $t \in \pi^{-1}(xq)$, i.e. suppose that $\pi(t) =_1 tq =_1 xq$. Thus, $t \bowtie xq$ holds. Opposite, let $s \bowtie xq$. Then $sq =_1 xq$, i.e. $\pi(s) =_1 sq =_1 xq$. So, $s \in \pi^{-1}(xq)$. Also, $\pi^{-1}(xq) = (xq)^C$ is a subsemigroup of S. The following implication

$$t \in \pi^{-1}(aq) \wedge y \in S \Longrightarrow ((ty\pi^{-1}(aq) \wedge yt \in \pi^{-1}(aq)) \vee t\theta y)$$

holds for every $a \in B$. To prove that, let $t \in \pi^{-1}(aq) \wedge y \in S$. Then $t \in \pi^{-1}(aq)) = (aq)^C$ and $y \in S$. Let u be an arbitrary element of aq. Then $(u, a) \in q$. Thus $(u, ty) \in q$ or $(ty, t) \in q$ or $(t, a) \in q$. Hence, $u \neq ty$ or $t\theta y$, because $(t, a) \in q$ is impossible. Analogously we show the implication $t \bowtie aq \wedge y \in S \Longrightarrow (yt \bowtie aq) \vee t\theta y)$.

(2) $\Longrightarrow$ (3). Let $(B, =, \neq, o, \leq_1, \theta_1)$ be a band and $f : S \longrightarrow B$ be a homomorphism such that $f^{-1}(b)$ is a subsemigroup of S for every $b \in B$. For each $b \in B$, we put $S_b = f^{-1}(b)$. Then we have the following:

(i) Let $a, b \in B$, $a \neq b$. Then $f^{-1}(a) \cap f^{-1}(b) = \emptyset$. Indeed: If $t \in f^{-1}(a) \cap f^{-1}(b)$, then $f(t) = a$, $f(t) = b$, so $a = b$ which is impossible.

(ii) $S = \cup_{a \in B} S_a$. Obviously, if $a \in B$, then $a \subseteq B$, so $f^{-1}(a) \subseteq S$ for every $a \in B$, and $\cup_{a \in B} S_a \subseteq S$. Let now $x \in S$. Then, for the element $a = f(x) \in B$, we have $x \in f^{-1}(a)$. So $x \in f^{-1}(a) = S \subseteq \cup_{b \in B} S_b$.

(iii) Let $a, b \in B$. Then $f^{-1}(a)f^{-1}(b) \subseteq f^{-1}(aob)$. Indeed, let $xy \in f^{-1}(a)f^{-1}(b)$, where $x \in f^{-1}(a)$, $y \in f^{-1}(b)$. Since f is a homomorphism, we have $f(xy) = f(x)of(y) = aob$, so $xy \in f^{-1}(aob)$.

(iv) Let $a, b \in B$ such that $S_a \cap (Sb] \neq \emptyset$. Then, $a \leq_1 b$ holds. Indeed, let $x \in Sa \cap (Sb]$. Since $x \in S_a = f^{-1}(a)$, we have $f(x) = a$. Since $x \in (Sb]$, there exists $y \in Sb$ such that $x \leq y$. Since $y \in S_b = f^{-1}(b)$, we have $f(y) = b$. On the other hand, since $x \leq y$ and f is order isotone homomorphism, we have $f(x) \leq_1 f(y)$. Therefore, $a \leq_1 b$.

(v) $t \in S_b \wedge y \in S \Longrightarrow ((ty \in S_b \wedge yt \in S_b) \vee t\theta y)$ for every $b \in S$. In fact: Let $t \in S_b = f^{-1}(b) \wedge y \in S$. By (2), we have

$$t \in \pi^{-1}(b) \wedge y \in S \Longrightarrow ((ty \in \pi^{-1}(b) \wedge yt \in \pi^{-1}(b)) \vee t\theta y)$$

i.e. we have

$$t \in S_b \wedge y \in S \Longrightarrow (ty \in S_b \wedge yt \in S_b) \vee t\theta y.$$

(3) $\Longrightarrow$ (1). Define a relation σ on S in the following way:

$$\sigma = \{(x, y) \in S \times S : (\exists a \in B)(y \in S_a \wedge x \in S_a)\}.$$

Then we have:

(I)σ is a band congruence on S. In fact,

If $x \in S(= \cup_{a \in B} S_q)$, then there exists $a \in B$ such that $x \in S_a$, thus $(x, x) \in \sigma$.

The relation σ is clearly symmetric.

Let $(x, y) \in \sigma$ and $(y, z) \in \sigma$. Then there exists $a \in B$ such that $x \in S_a \wedge y \in S_a$ and $b \in B$ such that $y \in S_b \wedge z \in S_b$. Then, $y \in S_a \cap S_b$. Thus, we have $S_a \cap S_b \neq \emptyset$ and, it has to be $S_a = S_b$. So, $x \in S_a \wedge y \in S_a$. Therefore, we have $(x, z) \in \sigma$.

Let $(x, y) \in \sigma$ and let z be an arbitrary element of S. Then, $(xz, yz) \in \sigma$. Indeed, since $(x, y) \in \sigma$, there exists $a \in B$ such that $x \in S_a \wedge y \in S_a$. Since $z \in S$, we have $z \in S_b$ for some $b \in B$. By hypothesis, we have $xz \in S_a S_b \subseteq S_{a \circ b}$, $yz \in S_a S_b \subseteq S_{a \circ b}$. Thus, $(xz, yz) \in \sigma$.

In a similar way we prove that σ is a left congruence on S.

Let $x \leq y$. Then $(x, xy) \in \sigma$ and $(x, yx) \in \sigma$. Indeed, since $x \in S$, there exists $a \in B$ such that $x \in S_a$. Since $y \in S$, there exists $b \in B$ such that $y \in S_b$. Since $x \in S \wedge x \leq y \in S_b$, we have $x \in (Sb]$. Hence, we have $x \in S_a \cap (S_b]$.

Then, by (3.4), we get $a \leq_1 b$ i.e. $a = aob = boa$. By (3.3), we have $xy \in S_a S_b \subseteq S_{a \circ b} = S_a$ and $yx \in S_b S_a \subseteq S_{b \circ a} = S_a$. Since $x \in S_a \wedge xy \in S_a (a \in B)$, we have $(x, xy) \in \sigma$. Also, $x \in S_a \wedge yx \in S_a (a \in B)$, imply $(x, yx) \in \sigma$.

Therefore, the relation σ is a band congruence on S.

(II) Let $x \in S$. Then $x\sigma$ is a subsemigroup of S. In fact; let $x \in S_a$ for some $a \in B$. We have $x\sigma = S_a$. Indeed: Let $y \in x\sigma$. Since $(y, x) \in \sigma$, there exists $b \in B$ such that $y \in S_b \wedge x \in S_b$. Then $x \in S_a \cap S_a$. Thus, we have $S_b = S_a$, $a = b$, and $y \in S_a$. Opposite, let $y \in S_a$. Since $x \in S_a \wedge y \in S_a$, where $a \in B$, we have $(x, y) \in \sigma$. Then $y \in y\sigma = x\sigma$.

(III) By [7], the relation $q = c(\sigma^C) = \cap_{n \in N}^{n} (\sigma^C)$ is a maximal coequality relation on semigroup S compatible with σ . Further on:

(i) Let $x \leq y$ and let (u, v) be an arbitrary element of q. Then $(u, x) \in q$ or $(x, xy) \in q$ or $(xy, v) \in q$ and $(u, x) \in q$ or $(x, yx) \in q$ or $(yx, v) \in q$. Thus, $(u, v) \neq (x, xy)$ and $(u, v) \neq (x, yx)$ because $(x, xy) \in q$ and $(x, yx) \in q$ are impossible by (I). So, $(x, xy) \bowtie q$ and $(x, yx) \bowtie q$.

(ii) Let x, y be arbitrary elements of S such that $(x, xy) \in q$. Then there exists a in B such that $x \in S_a$. Thus, by (3.5), we have $x \in S_a \wedge y \in S \Longrightarrow xy \in S_a \vee x\theta y$. If $xy \in S_a$, then $(x, xy) \in \sigma$. So, we have $x\theta y$ because the first case is impossible. Analogously, we have that the implication $(x, yx) \in q \Longrightarrow x\theta y$ holds.

(iii) Let $(ux, uy) \in q$ and let (s, t) be an arbitrary element of σ . Then $(us, ut) \in \sigma$ since σ is a congruence. Thus $(ux, uy) \in q \Longrightarrow (ux, us) \in q \vee (us, ut) \in q \vee (ut, uy) \in q \Longrightarrow ux \neq us \vee (us, ut) \in q \vee ut \neq uy \Longrightarrow x \neq s \vee t \neq y$ (because $\neg((us, ut) \in \sigma \wedge (us, ut) \in q)) \Longrightarrow (x, y) \neq (s, t) \in \sigma$.

Therefore, we have the implication
$(ux, uy) \in q \Longrightarrow (x, y) \bowtie \sigma$.

Let n be a natural number and suppose that the implication
$(ux, uy) \in q \Longrightarrow (x, y) \in^n \sigma$

holds. If (r, t) is an arbitrary element of ${}^{n+1}\sigma = \sigma *^n \sigma$, i.e. if

$$(\forall s \in S)((r, s) \in \sigma \vee (s, t) \in^n \sigma),$$

then $(\forall s \in S)((ur, us) \in \sigma \vee (us, ut) \in^n \sigma)$ and we have
$(ux, uy) \in q \Longrightarrow (\forall s \in S)((ux, ur) \in q \vee (ur, us) \in q \vee (us, ut) \in q \vee (ut, uy) \in q) \Longrightarrow ux \neq ur \vee (\forall s \in S)((r, s) \bowtie \sigma \vee (s, t) \bowtie^n \sigma) \vee ut \neq uy \Longrightarrow (x, y) \neq (r, t) \in^{n+1} \sigma$

because the second case is impossible. So, by induction, the formula $(\forall n \in \mathbf{N})((ux, uy) \in q \Longrightarrow (x, y) \bowtie^n \sigma)$ is valid. Thus, the implication $(ux, uy) \in q \Longrightarrow (x, y) \in \cap_{n \in N}^n (\sigma^C) = q$ holds. For another implication $(xu, yu) \in q \Longrightarrow (x, y) \in q$ the proof is analogous. Therefore, the relation q is a band anti-congruence on anti-ordered semigroup S. $\square$

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